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SET THEORY AND C* ALGEBRAS

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1. ULTRAPOWERS

Problem .1. *Does every separable C*-algebra embed into an ultrapower of $\mathcal{O}_2$ with respect to an ultrafilter on $\mathbb{N}$?*

Remark. [Kirchberg] Every exact C*-algebra embeds into $\mathcal{O}_2$.

If $\phi_j \colon A \to \prod_{\mathcal{U}} \mathcal{O}_2$ are *-homomorphisms, write $\phi_1 \leq \phi_2$ if there is a partial isometry u in A such that $u^* \phi_2 u = \phi_1$.

Problem .2. *Assume A is separable and A embeds into $\prod_{\mathcal{U}} \mathcal{O}_2$. Is there a $\leq$-maximal embedding ϕ of A into $\prod_{\mathcal{U}} \mathcal{O}_2$?*

Problem .3. *Can one prove in ZFC that for some free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ we have $\mathcal{B}(H)' \cap \prod_{\mathcal{U}} \mathcal{B}(H) = \mathbb{C}I$?*

An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is *flat* if there are $h_n \colon \mathbb{N} \searrow [0,1]$ such that

(1) $h_n(0) = 1$,
(2) $\lim_j h_n(j) = 0$,
(3) $(\forall f \colon \mathbb{N} \nearrow \mathbb{N}) \lim_{n \to \mathcal{U}} \sup_{j \in \mathbb{N}} |h_n(j) - h_n(f(j))| = 0$.

Problem .4. *Is a nonprincipal ultrafilter such that $\mathcal{B}(H)' \cap \mathcal{B}(H)^{\mathcal{U}} \neq \mathbb{C}I$ flat?*

2. CALKIN ALGEBRA

Problem .1. *Is there a model of set theory in which the Calkin algebra has an automorphism sending the unilateral shift S to S^*? Is the analogous fact for ℓ^∞ / c_0 true?*

Problem .2. *Is there a model of set theory in which the Calkin algebra has a not approximately inner automorphism preserving K-theory?*

Problem .3. *Is it consistent that, for a nonseparable Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ has an outer automorphism?*

Problem .4. *Does every masa in the Calkin algebra that is generated by its projections lift to a masa in $\mathcal{B}(H)$?*

3. Tensor products

Problem .1. *Does the number of C*-norms on $A \otimes_{alg} B$ for*
$A = B = \mathcal{B}(H)$
$A = B = C^*(\mathbb{F}_\infty)$, where $\mathbb{F}_\infty$ is the free group on countably many generators
$A = B(H)$ and $B = \mathcal{Q}(H)$ where $\mathcal{Q}(H)$ is the Calkin algebra
depend on the model of set theory?

Problem .2. *What is the ideal structure of $\mathcal{B}(H) \otimes_{\min} \mathcal{Q}(H)$? Does it depend on the model of set theory?*

Problem .3. *Does $\otimes_{\min}$ or $\otimes_{\max}$ commute with forcing?*

4. Nuclearity

Problem .1. *Is every stably finite nuclear separable C*-algebra locally approximable by type I C*-algebras?*

All presently known nuclear C*-algebras are obtained from $\mathbb{C}$ by taking closure under simple operations such as tensoring with finite-dimensional matrix algebras and taking inductive limits. They belong to the so-called bootstrap class.

Problem .2. *Does every nuclear C*-algebra belong to the bootstrap class?*

Problem .3. *Do all nuclear C*-algebras satisfy the UCT (Universal Coefficient Theorem)?*

5. Pure states

Problem .1. *Does every pure state of the atomic masa in $\mathcal{B}(H)$ extend uniquely to a pure state of $\mathcal{B}(H)$?*

Problem .2. *Consider the following statement: for every pure state ϕ of $\mathcal{B}(H)$ there is a masa $\mathcal{A}$ such that $\phi \restriction \mathcal{A}$ is multiplicative (i.e., pure). Is it consistent with ZFC?*

Problem .3. *Consider the following statement: every pure state on $\mathcal{B}(H)$ is diagonalizable. Is it consistent with with ZFC?*

6. Nonseparable C*-algebras

Problem .1. *Is there a nonseparable AF C*-algebra or W*-algebra not isomorphic to its opposite algebra?*

Problem .2. *Is the following relatively consistent with ZFC?*

Every C-algebra A that has a unique irreducible representation up to the unitary equivalence is isomorphic to the algebra of compact operators on some Hilbert space.*

Problem .3. *Assume A is a tensor product of algebras of the form $\mathbb{M}_n(\mathbb{C})$, for $n \in \mathbb{N}$, $\kappa < \kappa'$ are cardinals and $\bigotimes_{\kappa'} \mathbb{M}_2(\mathbb{C})$ unitally embeds into $A \otimes \bigotimes_\kappa \mathbb{M}_2(\mathbb{C})$. Can we conclude that there is a unital embedding of $\bigotimes_\kappa \mathbb{M}_2(\mathbb{C})$ into A?*

Problem .4. *Is every exact C*-algebra a subalgebra of a nuclear C*-algebra?*

Problem .5. *Is there a universal nuclear C*-algebra of character density $\aleph_1$? More generally, for which cardinals κ is there a universal nuclear C*-algebra of character density κ? Similar question can be asked for exact algebras.*

Problem .6. *Suppose a C*-algebra has an approximate identity consisting of projections. Does it have an increasing approximate identity (on some index set, possibly different) consisting of projections?*

7. Borel complexity

Problem .05. *Is conjugacy by automorphism of masas of the hyperfinite II_1 factor classifiable by countable structures?*

Problem .1. *What is the right set-theoretic framework to deal with functorial classification?*

Problem .15. *Is there a constructive Borel proof of the $\mathcal{O}_2$ embedding theorem?*

Problem .2. *Is the isomorphism relation in the following classes of separable C*-algebras:*
all
nuclear
exact
simple
simple exact
nuclear $\mathcal{Z}$-stable
simple nuclear

complete analytic? below a group actions? above all group actions?

Problem .25. *What is the complexity of isometric isomorphism of direct limits of not necessarily self-adjoint subalgebras of finite dimensional C*-algebras? How does it compare to the complexity of isomorphism of AF algebras?*

Problem .3. *Is there a Borel inverse of the classification functor?*

Problem .35. *For a separable C*-algebra A, what is the complexity of orbit equivalence relations associated to the action* $\mathrm{Aut}(A)$ *on itself by conjugacy and to the action of* $\mathrm{Inn}(A)$ *on* $\mathrm{Aut}(A)$ *by left translation? Are these action turbulent? How are they related to structural properties of A?*

Problem .4. *Is the Mackey Borel structure on the spectrum of a simple separable C*-algebra always the same when it is not standard?*

Remark. [Elliott] All nonstandard spectra of AF algebras are isomorphic.

Problem .45. *Does the complexity of the Mackey Borel structure of a simple separable C*-algebra increase as one goes from nuclear C*-algebras to exact ones to ones that are not even exact?*

If A is a C*-algebra, denote by E_A the relation of unitary equivalence of pure states of A.

Problem .5. *Assume A and B are C*-algebras and* E_A *is Borel-reducible to* E_B*. What does this fact imply about the relation between A and B?*

8. Generators

A C*-algebra is called singly generated if it contains an element that is not contained in any proper sub-C*-algebra. A C*-algebra is called $\mathcal{Z}$-stable if it absorbs the Jiang-Su algebra tensorially.

Problem .1. *Which separable, unital, simple C*-algebras are singly generated? Is there a simple, nuclear C*-algebra that is singly generated, yet is not* $\mathcal{Z}$*-stable? Is single generation connected to the regularity properties coming up in the classification of nuclear, simple C*-algebras?*

References