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MANIFOLDS WITH NON-NEGATIVE SECTIONAL CURVATURE

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1. Diameter Pinching

Problem 1.1. *Does the Grove-Petersen-Wu Finiteness Theorem hold in dimension* 4*, i.e. are there only finitely many diffeomorphism types in the class of* 4*-dimensional Riemannian manifolds satisfying*

$$\sec_M \geq -\Lambda^2, \ \ \operatorname{diam}(M) \leq D, \ \ \operatorname{Vol}(M) \geq V?$$

Problem 1.2. *If* $\sec_M \geq 1$ *and* $\operatorname{diam}(M) > \frac{\pi}{2}$*, must* M *be diffeomorphic to* S^n*?*

Problem 1.3. *Let* M^n *be a manifold for which* $\sec_M \geq 1$ *and* $\operatorname{diam}(M) \geq \frac{\pi}{2} - \varepsilon(n)$*. Is* M *homeomorphic to a manifold* M'*, where* $\sec_{M'} \geq 1$*,* $\operatorname{diam}(M') \geq \frac{\pi}{2}$*? This problem may be easier to address if we also assume that* $\operatorname{Vol}(M) \geq V$ *and* $\varepsilon = \varepsilon(n, V)$*.*

2. Collapse and Alexandrov Geometry

Problem .05. *Perelman's Stability Theorem yields that manifolds in a given sequence of non-collapsing manifolds are eventually pairwise homeomorphic. Are they also PL-homeomorphic or diffeomorphic?*

Problem .1. *Understand DC-structures on manifolds. In particular, does Perelman's Stability Theorem hold in the DC-category. Is PL = DC always?*

Problem .15. *Extend the Wilking Connectivity Theorem to Alexandrov spaces, i.e. if* X *is a positively curved Alexandrov space and* $Y \subset X$ *a totally geodesic subspace of codimension* k*, is it true that* $X - Y$ *has homology only up to dimension* $2k - 2$*?*

Problem 2.2. *Suppose* X *is the non-collapsed Gromov-Hausdorff limit of* (M_i^n, g_i)*, where* $|\sec_{M_i}| \leq 1$*, and that along every geodesic on* M_i *one hits a conjugate point before* $t = \pi + \frac{1}{i}$*. Is* X *rigid in any sense?*

Problem 2.25. *Is there a sequence of simply-connected, pointwise strictly* $\frac{1}{4}$*-pinched manifolds* M_i^n*,* $n > 2$*, that collapse?*

Problem 2.3. *Find an appropriate definition of Morse functions on Alexandrov spaces and construct examples.*

Problem 2.35. *Study the collapse of Alexandrov spaces.*

Consider finite towers

$$M_0[r]^{F_1}M_1[r]^{F_2}\dots[r]^{F_k}M_k \tag{2.1}$$

of fiber bundles, where the fibers $\{F_1,\dots,F_k\}$ and M_k are fixed topological manifolds.

Problem 2.4. *Loosen this notion to get a "brotherhood" on the manifolds M_0 and a property of such M_0 not known to be possessed by all manifolds of* $\sec \geq K$, $\operatorname{diam} \leq 1$.

Problem 2.45. *Give an Alexandrov analogue of rational ellipticity. In particular, are manifolds with almost non-negative sectional curvature rationally elliptic?*

Problem 2.5. *Given a non-collapsing Gromov-Hausdorff convergence $M_i \longrightarrow X$, can one find a "tangent bundle" structure on X that is sensitive to the diffeomorphism class of the M_i?*

Problem 2.55. *Is there a Gauss formula for Alexandrov spaces, i.e. must a convex hypersurface Y of an Alexandrov space X have* $\sec_Y \geq \sec_X$*?*

Problem 2.6. *Is every finite dimensional Alexandrov space a limit of Riemannian manifolds with* $\sec \geq K$*?*

Problem 2.65. *Study Alexandrov (almost) submetries.*

Problem 2.7. *Is there an alternate approach to homotopy groups that is adapted to Alexandrov spaces?*

Problem 2.75. *Study collapse to a ray.*

Problem 2.8. *Can an n-dimensional torus collapse to an interval?*

Problem 2.85. *Study the collapse of Riemannian manifolds with boundary which have* $\sec \geq K$ *on the interior and controlled boundary concavity.*

Problem 2.9. *Find an application where infinite-dimensional Alexandrov spaces appear as limits of manifolds of increasing dimension.*

3. Group Actions and Submersions

Problem 3.05. *Let M^n be a manifold with* $\sec > 0$ *or* $\sec \geq 0$ *or almost non-negative curvature. Does M^n have a positive symmetry degree, i.e. is there an $S^1 \subset \mathrm{Diff}(M^n)$?*

Problem 3.1. *Is there a principal T^2-bundle whose total space admits* $\sec > 0$*?*

Problem 3.15. *Given a fat G-principal bundle, must G be S^1, S^3 or $SO(3)$?*

Problem 3.2. *Can one reduce the structure group of a fat principal G-bundle?*

Problem 3.25. *Given a homogeneous space G/H, with G compact, classify all homogeneous metrics with* $\sec \geq 0$*.*

Problem 3.3. *Is there a positively curved* 5*-manifold with a free isometric S^3 or $SO(3)$ action?*

Problem 3.35. *Given a Riemannian submersion with positively curved total space, is the dimension of the fiber less than the dimension of the base? It is perhaps simpler to decide if there is a bound on the dimension of the fiber in terms of the dimension of the base. Is the image of the A-tensor large at some point?*

Problem 3.4. *Classify Riemannian submersions from a Lie group with a bi-invariant metric.*

Problem 3.45. *Suppose (M^n, g) is simply-connected and* $\sec_M \geq 0$*. Is* $\mathrm{rank}(Iso(M^n, g)) \leq \frac{2}{3}n$*?*

Problem 3.5. *Suppose we have an isometric group action on M. If one changes the metric on the orbit space, does it lift to an invariant metric on M?*

Problem 3.55. *If a group acts isometrically on a manifold of* $\sec \geq 1$ *and the fixed point set is a circle, is it's length* $\leq 2\pi$*?*

4. Manifolds of Cohomogeneity-one and Polar Actions

Problem 4.3. *Classify cohomogeneity-one manifolds with* $\sec \geq 0$ *and at least one totally geodesic principal orbit.*

Problem 4.4. *Compute topological invariants of cohomogeneity-one manifolds. Classify topologically the new candidates for positive curvature.*

Problem 4.5. *Study the existence of Einstein metrics on manifolds of cohomogeneity-one.*

Problem 4.6. *Suppose M is a polar manifold with* $\sec > 0$*. Must M be diffeomorphic to a compact rank-one symmetric space?*

Problem 4.7. *Suppose $\Sigma \subset M^n$ is the section of a polar action. If Σ is rationally elliptic, must M^n be rationally elliptic?*

Problem 4.2. *Find cohomogeneity-one manifolds with "interesting" topology, in particular not homeomorphic to a symmetric space. Study curvature properties of these manifolds.*

Problem 4.1. *Study the existence and non-existence of metrics with non-negative curvature on manifolds of cohomogeneity-one.*

5. Vector Bundles

Problem 5.1. *Is there a metric with* $\sec \geq 0$ *on $\mathbb{R}^6$-bundles over $S^3 \times S^3$ with non-trivial Euler class? Wilking has shown that the answer is "No" if the soul is $S^3 \times S^3$ with the product metric.*

Problem 5.2. *Which vector bundles over $S^2 \times S^2$ or $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$ where the structure group does not reduce to a torus admit* $\sec \geq 0$*?*

Problem 5.3. *Classify metrics with* $\sec \geq 0$ *on $S^2 \times \mathbb{R}^4$ (or, more generally, on $S^n \times \mathbb{R}^k$).*

Problem 5.4. *Suppose $E \longrightarrow M$ is a vector bundle over a compact, simply-connected manifold M for which* $\sec_M \geq 0$*. Does $E \oplus \mathbb{R}^k \longrightarrow M$ have* $\sec \geq 0$ *for k large?*

6. Quasi-positive Curvature and Positive Curvature on an Open Dense Set

Problem 6.1. *Which theorems from* $\sec > 0$ *carry over to positive curvature on an open dense set?*

Problem 6.2. *Suppose G is a compact Lie group with a left-invariant metric. Are there any new examples $H\backslash G$ with quasi-positive curvature?*

Problem 6.3. *Find new examples of fundamental groups in quasi-positive curvature or positive curvature on an open dense set.*

Problem 6.4. *Fix $k \in \mathbb{N}$. Is there a $n_0 = n_0(k)$ such that for any quasi-positively curved manifold (M^n, g) with $n \geq n_0$ and $\operatorname{cohom}(M^n, g) \leq k$, there exists a chain*

$$M_0 = M^n \subset M_1^{n+k} \subset M_2^{n+2k} \subset \cdots$$

such that all inclusions are totally geodesic, the manifolds M_i are quasi-positively curved, $\cup M_i$ is the classifying space of a Lie group, and $M_i/Iso(M_i, g)$ is isometric to $M/Iso(M, g)$?

7. Ricci Flow

Problem 7.1. *Let M^n be a compact manifold with positive isotropic curvature. Does any blow-up limit of the Ricci flow have non-negative curvature operator?*

Problem 7.2. *Suppose G acts freely and isometrically on M. What kind of flows on M/G are induced by the Ricci flow on M?*

Problem 7.3. *Can one improve the Hsiang-Kleiner theorem on positively curved 4-manifolds with symmetry from homeomorphism to diffeomorphism by using the Ricci flow?*

8. Miscellaneous Problems

Problem 8.1. *Is a "generic" manifold a $K(\pi, 1)$-space (where "generic" is to be determined)?*

Problem 8.2. *Does positive sectional curvature imply that the manifold is formal (in the sense of Sullivan's minimal model)?*

Problem 8.3. *For compact, odd dimensional, positively curved manifolds is there a cyclic subgroup of the fundamental group whose index is bounded only in terms of the dimension?*

Problem 8.4. *Is there a finiteness result for n-dimensional, positively curved manifolds with $\pi_1 = \pi_2 = 0$?*

Problem 8.5. *Is there a $\delta(n) > 0$ such that any n-dimensional, positively curved manifold carries a $\delta(n)$-pinched metric?*

References