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MAPPING THEORY IN METRIC SPACES

EDITED BY

1. Extensions

Linear dependence of constants in bi-Lipschitz extension theorems

Problem .1. *Do bi-Lipschitz extension theorems admit linear dependence on constants?*

Remark.[Leonid Kovalev] For bi-Lipschitz embeddings of the circle, the answer is yes for extension to the disk, and no for extension to the plane. Reference: Kovalev, L.V., ”Optimal extension of Lipschitz embeddings in the plane”, Bull. London Math. Soc. 51 (2019), no. 4, 622-632.

Dependence on parameters

Problem .2. *Do canonical Lipschitz extensions exist which depend nicely on parameters?*

Deterministic proof of the Lee–Naor Lipschitz extension theorem

The Lee–Naor theorem states that, given a doubling subset A of a metric space X and an L-Lipschitz map from A to a Banach space Y, there exists a $C(\log m)L$-Lipschitz extension $X \to Y$. Here m is the doubling constant of A.

Problem .3. *Is there a deterministic proof of the Lee–Naor Lipschitz extension theorem giving the sharp dependence on the doubling constant?*

Remark. There are deterministic proofs of similar results by LangSchlichenmaier and Brudnyi-Brudnyi.

Absolute Lipschitz retract constructions

$\mathbb{Q}_Q(Y)$ denotes the space of unordered Q-tuples $y = [y_1, \ldots, y_Q]$ of elements of Y equipped with the metric $d(y,z) = \min\{\max\{d(y_j, z_{\sigma(j)})\} : \sigma \in S_Q\}$, where S_Q denotes the symmetric group on Q letters.

Problem .4. *If Y is an absolute Lipschitz retract, is the same true of the space $\mathbb{Q}_Q(Y)$?*

Remark.[Leonid Kovalev] The space $\mathbb{Q}_Q(Y)$ is similar to the Qth symmetric product space; similar questions could be of interest for symmetric products.

Banach space pairs with the Lipschitz extension property

Problem .5. *Does the pair (L^2, L^1) have the Lipschitz extension property?*

Absolute Lipschitz retracts

Problem .6. *Are all Hadamard manifolds absolute Lipschitz retracts?*

Lipschitz homotopy groups of the Heisenberg group

$\mathbb{H}^n$ denotes the nth Heisenberg group, equipped with the Carnot-Carathéodory metric. $\pi_k^{Lip}(X)$ denotes the kth Lipschitz homotopy group of a metric space X. It is known that $\pi_n^{Lip}(\mathbb{H}^n) \neq 0$ (DeJarnette–Hajlasz–Lukyanenko–Tyson) [**?**].

Problem .7. *If $\pi_k(\mathbb{S}^n)$ is nontrivial, is the same true for $\pi_k^{Lip}(\mathbb{H}^n)$? In particular, is $\pi_3^{Lip}(\mathbb{H}^2) \neq 0$?*

Remark. Solved by Hajlasz, Tyson, Schikorra, Wenger, Young... in HOMOTOPY GROUPS OF SPHERES AND LIPSCHITZ HOMOTOPY GROUPS OF HEISENBERG GROUPS as well as "Lipschitz Homotopy Groups of the Heisenberg Groups", to appear in GAFA (Geometric and Functional Analysis) in February 2014.

Lipschitz homotopy groups of the Heisenberg group II

Problem .8. *Does $\pi_n^{Lip}(\mathbb{H}^n)$ contain torsion elements?*

2. EMBEDDINGS

Deterministic proofs of bi-Lipschitz embedding theorems

Assouad's embedding theorem asserts that to each doubling metric space (X, d) and each $\epsilon \in (0, 1)$, there corresponds n so that the snowflake metric space (X, d^ϵ) admits a bi-Lipschitz embedding into $\mathbb{R}^n$. Naor–Neiman proved that n can be chosen independent of the snowflake parameter ϵ, for ϵ near one (say, $\epsilon > \frac{1}{2}$). Their proof is probabilistic and nonconstructive. Can one give an explicit construction of such embedding?

Problem .1. *Give a deterministic proof of Naor–Neiman's improved Assouad embedding theorem.*

Remark. Solved by Guy David and Marie Snipes in ”A Non-Probabilistic Proof of the Assouad Embedding Theorem with Bounds on the Dimension.” Analysis and Geometry in Metric Spaces 1 (2013): 36-41. ¡http://eudml.org/doc/267210¿

Curvature conditions and bi-Lipschitz embeddings

Problem .3. *Let X be a doubling metric space. Can one formulate a curvature-type condition which, if satisfied by X, ensures that X admits a bi-Lipschitz embedding into some finite-dimensional Euclidean space?*

Bi-Lipschitz embeddings: Hilbert spaces vs. finite-dimensional Euclidean spaces

Problem .4. *Suppose that X is a doubling space which admits a bi-Lipschitz embedding into Hilbert space. Does X necessarily admit a bi-Lipschitz embedding into some finite-dimensional Euclidean space?*

Snowflake embeddings of the Heisenberg group into low-dimensional Euclidean spaces

Let d_{cc} denote the Carnot-Carathéodory metric.

Problem .4. *Does there exist* $\epsilon < 1$ *so that the first Heisenberg group* $\mathbb{H}^1$ *equipped with the metric* d_{cc}^{ϵ} *bi-Lipschitz embeds into* $\mathbb{R}^5$*?*

Remark. Instead one could use the Korányi metric $d_0(p,q) = \|p^{-1} * q\|_0$, where $\|(z,t)\|_0 = (|z|^4 + t^2)^{1/4}$.

3. Uniformization and parameterization

Bi-Lipschitz parameterization by Euclidean spaces

Problem 3.1. *Are there higher dimensional analogs of the Bonk–Lang parameterization theorem which give quantitative control on bi-Lipschitz constants?*

Quasisymmetric uniformization of metric 2*-spheres*

Problem 3.2. *Give a geometric characterization of metric* 2*-spheres which are quasisymmetrically equivalent to the standard* 2*-sphere.*

Remark. See also: https://link.springer.com/article/10.1007/s00229-012-0555-0

Bi-Lipschitz parameterization of metric 2-spheres

Problem 3.3. *Give a geometric characterization of metric 2-spheres which are equivalent to the standard 2-sphere via a bi-Lipschitz map.*

4. REGULARITY

Almost everywhere partial differentiability of mappings of integrable lower metric distortion

The lower metric dilatation of a homeomorphism $f : X \to Y$ of metric spaces is defined as

$$h_f(x) = \liminf_{r\to 0} \frac{\sup\{d(f(x), f(y)) : d(x,y) \le r\}}{\inf\{d(f(x), f(z)) : d(x,z) \ge r\}}.$$

Problem 4.1. *Let f be a homeomorphism of planar domains with $h_f \in L^1_{loc}$ and h_f finite off a set of σ-finite length. Does f have partial derivatives almost everywhere?*

Mapping properties of planar maps of exponentially integrable distortion

Problem .2. *Does there exist a homeomorphism f of the plane with exponentially integrable distortion such that f fixes the real axis and sends a set of positive length onto the $\frac{1}{3}$ Cantor set?*

Approximation of Sobolev homeomorphisms by diffeomorphisms

Problem .3. *Let f be a homeomorphism of the plane such that f and f^{-1} lie in some Sobolev class, e.g., $W^{1,2}$. Can one approximate f by diffeomorphisms f_j so that f_j and f_j^{-1} converge in the Sobolev norm?*

Remark. The one-sided approximation, namely $f_j \to f$ without the convergence of inverses, is possible in this case (Iwaniec–Kovalev–Onninen).

Remark. Even the one-sided case remains open in the non-reflexive space $W^{1,1}$, or for any Sobolev space in dimension three.

5. RIGIDITY

Rigidity of n-harmonic functions in $\mathbb{R}^n$

Problem .1. *Is there an elementary proof for the fact that every entire n-harmonic function in $\mathbb{R}^n$ with linear growth is affine?*

Are all 1-quasiconformal homeomorphisms of Hilbert space similarities?

Problem .2. *Let f be a homeomorphism between Hilbert spaces with $H_f = 1$ everywhere. Is f a similarity?*

Remark. This problem is likely due to Jussi Väisälä.

Remark. One could start by asking for differentiability results for quasisymmetric homeomorphisms between Hilbert spaces.

Quasiconformal mappings of the plane which destroy the rectifiability of uncountably many disjoint lines

Problem .3. *Is there a quasiconformal map f of the plane and an uncountable set $E \subset \mathbb{R}$ so that $f(E \times \mathbb{R})$ contains no rectifiable curves?*

Remark. The same question may also be interesting for mappings of exponentially integrable distortion.

Quasisymmetric maps of products of irreducible Carnot groups

Problem .4. *Let H be an irreducible Carnot group, G a product of several copies of H, and $f : G \to G$ a quasisymmetric map. Must f be a product map (after permutation of the factors)?*

Product quasiconformal mappings of the plane

Problem .5. *Let f be a quasiconformal map of the plane. Suppose at a.e. point, the differential of f either preserves both the x-direction and the y-direction, or switches the two directions. Does this imply f is a product map?*

Loewner Sierpiński carpets

Problem .6. *Is the usual $\frac{1}{3}$ Sierpiński carpet S_3 quasisymmetrically equivalent to a Loewner space?*

Remark. This problem is closely related to the well known open problem of determining the conformal dimension of S_3.

Bi-Lipschitz embeddings of the Sierpiński carpet into itself

Problem .7. *Is every bi-Lipschitz embedding of the $\frac{1}{3}$ Sierpiński carpet S_3 into itself the restriction of an affine mapping?*

Quasisymmetric rigidity and non-rigidity of positive area Sierpiński carpets

A set $X \subset \mathbb{R}^n$ is said to be *quasisymmetrically rigid* if every quasisymmetric map of X onto itself coincides the restriction to X of an isometry of $\mathbb{R}^n$. Bonk and Merenkov have shown that the usual $\frac{1}{3}$ Sierpiśnki carpet is quasisymmetrically rigid.

Problem .8. *Which positive area Sierpiński carpets are quasisymmetrically rigid? Which are nonrigid?*

6. Variational problems

Inner variations

Consider the inner variational equation for planar maps

$$(f_z \overline{f_{\bar{z}}})_{\bar{z}} = 0.$$

Locally, this equation can be reduced to a first-order equation $f_z f_{\bar{z}} = 1$, which may also be viewed as a differential inclusion $Df \in M$, with M a certain set of 2×2 matrices. Natural assumptions on f are the finiteness of the energy and nonnegativity of the Jacobian.

The equation

$$(f_z \overline{f_{\bar{z}}})_{\bar{z}} = 0.$$

expresses the stationarity of the Dirichlet energy $\int |Df|^2$ under inner variations of f (precomposition with diffeomorphisms).

Problem 6.1. *Is $|Df|$ continuous?*

Variational problems for dilatation functionals

Problem .2. *Study extremal variational problems for dilatation functionals of mappings. For instance, find the extremal domain (and possibly also the extremal mappings) associated to the configuration of a doubly connected planar domain including an obstacle of prescribed diameter.*

Hausdorff measures in ℓ_n^p

Let $\mathcal{H}^m$ denotes the Hausdorff m-measure corresponding to the norm in ℓ_n^p.

Problem .3. *Let P be a polyhedral m-cycle in ℓ_n^p, $p \neq 2$, with faces $F_1, \ldots, F_k$. Is $\mathcal{H}^m(F_1) \leq \sum_{j=2}^k \mathcal{H}^m(F_j)$?*

Compact deformations of minimizing sets

Problem .4. *An m-rectifiable closed set $S \subset V = \ell_n^\infty$ is called minimizing if for every Lipschitz map $f : V \to V$ such that f is the identity off a cube C and $f(C) \subset C$, $\mathcal{H}^m(C \cap S) \leq \mathcal{H}^m(f(C \cap S))$. Let S be minimizing. Is it true that for $\mathcal{H}^m$-almost every $x \in S$, there exists a neighborhood U of x such that $S \cap U$ is a Lipschitz graph over an m-dimensional subspace Π of V?*

Isoperimetric inequalities in metric measure spaces

Problem .5. *Can the perimeter measure P on the right hand side of the relative isoperimetric inequality*

$$\min\{\mu(E \cap B), \mu(B \setminus E)\} \leq CrP(E, \lambda B),$$

where $E \subset X$ is a Borel set and $B \subset X$ is a ball of radius r, be replaced with the codimension one Hausdorff measure of the part of the measure-theoretic boundary of E inside B even without knowing ahead of time whether E is of finite perimeter?

References