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KARDAR-PARISI-ZHANG EQUATION AND UNIVERSALITY CLASS

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We list here open problems, which were discussed during the open problems sessions of the workshop ”KPZ equation and universality class” at AIM, September 2011, organized by Ivan Corwin and Jeremy Quastel.

1. BIG PICTURE QUESTIONS

We have organized these based on perceived level of difficulty within each category.

Statistics of KPZ equation in $1+1$ dimension:

Problem 1.05. *Compute statistics for different initial data including $\mathcal{Z}_0(x) = 1$ (flat) and $\mathcal{Z}_0(x) = e^{B(x)}$ for $B(x)$ a two-sided Brownian motion (equilibrium).*

Problem 1.1. *Compute multi-point (spatial) distribution for KPZ with various initial data. For example, with $\mathcal{Z}_0(x) = \delta_{x=0}$, let*

$$\mathcal{Z}(t,x) = e^{-\frac{x^2}{2t}+t^{1/3}A_t(t^{-2/3}x)-\frac{t}{24}}.$$

What is the distribution $F(\xi_1,\xi_2)$ such that

$$\mathbb{P}\left(A_t(x_1)\le\xi_1, A_t(x_2)\le\xi_2\right) = F(\xi_1,\xi_2)?$$

Perhaps easier, show that as t goes to infinity and space is scaled like $t^{2/3}x$, the process $A_t(t^{2/3}x)$ converges to the Airy$_2$ process in x.

Problem 1.15. *Compute multi-time distribution for KPZ. What is the distribution $F'_{t_1,t_2}(\xi_1,\xi_2)$ such that*

$$\mathbb{P}\left(A_{t_1}(x)\le\xi_1, A_{t_2}(x)\le\xi_2\right) = F'_{t_1,t_2}(\xi_1,\xi_2)?$$

Higher dimensions

Problem 1.2. *Make rigorous sense of the anisotropic KPZ equation*

$$\partial_t h = (\partial_x h)^2 - (\partial_y h)^2 + \Delta h + \xi$$

and show that the Gaussian free field is invariant. Is there a way of getting this equation out of the 2d Schur process dynamics of Borodin-Ferrari? Or perhaps out of the 2d q-Whittaker process dynamics of Borodin-Corwin? Is this related to 2d quantum Toda chain?

Problem 1.25. *Prove that above 2 spatial dimensions, the KPZ equation is trivial (i.e., discretizations limit to the linearized equation, perhaps with a larger variance in the noise).*

Problem 1.3. *In 2 spatial dimensions is the measure valued solution of the stochastic heat equation related to the "exponential" of the dynamic GFF and hence quantum Louiville gravity?*

Problem 1.35. *Study polymer free energy and growth model fluctuation exponents in higher dimension? Compute limit shapes and fluctuation exponents.*

Structure at positive temperature / asymmetry:

Positive temperature polymers have triangular arrays associated with them (which are limits of the Macdonald processes). The marginal measure on a given level is a tropical analog of random matrix eigenvalue ensembles (such as GUE or LUE) which are determinantal point processes. These tropical point processes are no longer determinantal yet, there are still Fredholm determinants for Laplace type transforms of the analog of the largest or smallest eigenvalues. Information about all of the tropical eigenvalues should be accessible via Fredholm determinants (tropical analogs of gap probabilities for example).

Problem 1.4. *What is the structure which replaces determinantal point processes and correlation functions and which allows for such computations?*

Associated to q-TASEP there is a triangular array (the q-Whittaker process). This whole array is helpful in computing things about q-TASEP.

Problem 1.45. *Is there such an array for ASEP or for some sort of transformed version of ASEP?*

The limit of the triangular array is the diffusion of O'Connell based on the quantum Toda lattice, or (after another limit) the KPZ_T line ensemble. Both have Brownian Gibbs properties allowing paths to cross but at exponential cost (a soft analog of the non-intersecting Brownian Gibbs property for the Airy line ensemble.)The Karlin-McGregor formula underlies the solvability of zero temperature system since it writes non-intersecting line ensembles in terms of determinants.

Problem 1.5. *Is there an analog of the Karlin-McGregor formula when the non-intersecting conditioning is replaced by a softer form of conditioning? Does this explain the solvability and occurrence of Fredholm determinants?*

Expand the universality of the KPZ equation to:

Problem 1.55. *Growth model such as ballistic deposition.*

Problem 1.6. *Height functions associated to exclusion processes with longer range jumps, or environment dependent speed-changes.*

Problem 1.65. *Stochastic Hamilton-Jacobi equations* $\partial_t h = F(\nabla h) + \Delta h + \xi_\epsilon$ *with* F *scaled appropriately.*

Problem 1.7. *Eden model.*

Universality of the KPZ universality class:

Problem 1.75. *Prove the existence and uniqueness of the KPZ universality class fixed point introduced by Corwin-Quastel in 1 spatial dimension. That is to say, consider any growth process* $h(t, x)$ *and show that for an appropriate centering function* $\overline{h}_\epsilon$*,*

$$\lim_{\epsilon\to\infty} \epsilon^{1/2} h(\epsilon^{-3/2} t, \epsilon^{-1} x) - \overline{h}_\epsilon$$

has a limit as $\epsilon \to 0$ *as a space-time process? Then show that the properties of this process identify it uniquely (see the conjectured properties in Corwin-Quastel).*

Problem 1.8. *Prove that the fixed point is attractive (i.e., universal in some class of models).*

Problem 1.85. *For example, prove universality of LPP and polymer (with respect to weight distributions).*

Problem 1.9. *Provide a variational explanation for the actual form of the GUE Tracy-Widom distribution in terms of the properties of the fixed point. Reduce the exact solvability to purely probabilistic terms.*

2. Open problems

Exact solvability

Problem 2.02. *Use the Macdonald processes formulas to prove the Baik-Ben Arous-Péché transition for the semi-discrete polymer with a finite number of non-zero drifts for the underlying Brownian motions (tuned critically). In the intermediate disorder scaling, derive the formula for the statistics of the stochastic heat equation started with* $\mathcal{Z}_0(X) = \mathbf{1}_{X\geq 0} Z^r(X)$ *where* $Z^N(X)$ *is the partition function for the semi-discrete polymer with r levels at time X (the r here represents the number of spikes in BBP and the drifts here of the r Brownian motions can be generally chosen).*

Stochastic analysis

Problem 2.2. *Use the well-posedness theory for the KPZ equation to redo Bertini-Giacomin or to prove convergence of simpler discretizations of the KPZ equation such as the one previously studied by Sasamoto and Spohn.*

Exact solvability

Problem 2.06. *Motivated by success in solvable positive temperature polymers, try to study the polynuclear growth model with an underlying Brownian path measure – is there a Burkes theorem in this setting? Specifically, study the Poissonian last passage model but at positive temperature and with the Poisson points having random weights.*

Exact solvability

Problem 2.08. *Study the overlap for replicated paths of a polymer with respect to the quenched Gibbs measure. What is the distribution of the intersection local time for the continuum polymer? Is there replica symmetry or replica breaking in the strong disorder regime or intermediate disorder regime? Try to use the Macdonald processes contour integral formulas to compute more than just the expected overlap.*

Stochastic analysis

Problem 2.1. *Find a microscopic Gärtner transform (discrete Hopf-Cole transform) which turns the dynamics of q-TASEP into a discrete stochastic heat equation or better yet a polymer. Then prove this converges to the continuum stochastic heat equation.*

Interacting particle systems

Problem 2.12. *Bulk of q-Whittaker* 2*d dynamics. Invariant measure.*

Interacting particle systems

Problem 2.14. *Study q-TASEP hydrodynamics and other interacting particle systems properties such as second class particles, invariant distributions, and Burkes type theorem.*

Interacting particle systems

Problem 2.16. *What about q-ASEP, where particle can jump to right and left and feel a repulsion from their right or left neighbors: Is there Burkes Theorem? Do second class particle methods work?*

Gibbs line ensembles

Problem 2.18. *Consider a Brownian excursion on time interval $[-N, N]$ condition on fixed area N. Take $1/3, 2/3$ scalings around the edge. Do we get an "Airy-like" limiting fluctuation?*

Polymer universality

Problem 2.4. *Determine limiting behavior of polymer without a sixth moment, and with $\beta \to 0$ at the correct rate to have a limit.*

Polymer universality

Problem 2.22. *Consider intermediate scaling for polymer with $\beta = n^{-1/4}c(n)$ where $c(n)$ is any function which tends to infinity with n. Show universality of the F_{GUE} fluctuations with a fluctuation exponent which may depend on $c(n)$.*

Remark. This requires careful estimation of terms in the discrete chaos series which are further and further out as n increases (reminiscent of Soshnikovs approach to edge universality for Wigner matrices).

Exact solvability

Problem 2.26. *Investigate the relationship between the tropical RSK correspondence and mirror symmetry: The entrance law for the Markov process which arises from adding columns under the tropical RSK correspondence arises via a critical point calculation.*

Remark. The critical point has also arose in the study of mirror symmetry.

Universality of initial data for (T)ASEP.

Problem 2.3. *Consider two initial conditions for (T)ASEP corresponding to height functions* $h_1(x; t = 0)$ *and* $h_2(x; t = 0)$ *which can be random, but are independent of each other. Hydrodynamic theory says that if for* $i = 1, 2$, $\epsilon^{-1} h_i(\epsilon^{-1} x; t = 0) \to \bar{h}(x; t = 0)$ *as* $\epsilon \to 0$, *then so does* $\epsilon^{-1} h_i(\epsilon^{-1} x; t) \to \bar{h}(x; t)$ *where* $\bar{h}$ *solves a Hamilton-Jacobi conservation law with quadratic flux. We would like a similar result, but for fluctuations. This result would show that if we assume for* $i = 1, 2$,

$$\epsilon^{1/2} \left[h_i(\epsilon^{-1} x; t = 0) - \epsilon^{-3/2} \bar{h}(\epsilon^{1/2} x; 0) \right] \to \tilde{h}(x; t = 0)$$

as $\epsilon \to 0$, *then so does*

$$\epsilon^{1/2} \left[h_i(\epsilon^{-1} x; \epsilon^{-3/2} t) - \epsilon^{-3/2} \bar{h}(\epsilon^{1/2} x; t) \right] \to \tilde{h}(x; t).$$

Universality of initial data for (T)ASEP.

Problem 2.32. *Show* $T^{1/3}$ *fluctuations for KPZ equation slightly out of equilibrium by using a variant of the second class particle method for WASEP near equilibrium.*

Gibbs line ensembles

Problem 2.34. *Show tightness and limiting Gibbs property for line ensembles besides the Airy line ensemble (e.g. Bessel process, Peary process, Sine process).*

Exact solvability

Problem 2.36. *Explain the occurrence of a boundary value problem (BVP) in the kernel for the continuum statistics for the Airy*$_2$ *process:* $\mathbb{P}(A_2(\cdot) \leq g(\cdot)) = \det(I - M)$, *where* M *has a kernel given in terms of a BVP involving* g. *Is this due to the Karlin-McGregor formula for this infinite ensemble of lines? Study the analogous continuum statistics for the top curve of N non-intersecting Brownian bridges or Brownian motions. This might explain where this BVP comes from.*

Exact solvability

Problem 2.38. *Study solvable variants of the log-Gamma polymer with different types of polymer paths.*

Stochastic analysis

Problem 2.04. *Consider the multi-layer extension of the stochastic heat equation introduced by O'Connell and Warren. It is defined via chaos series – show that it can also be defined by spatially smoothing white-noise and then making sense of the Wick exponential in its formulation as a partition function for non-intersecting Brownian bridges in a space time white-noise environment.*

REFERENCES