[2007/11/11]

STABILITY AND HYPERBOLICITY

EDITED BY

A list of the problems from the workshop on stability, hyperbolicity, and zero localization of functions, held December 5-9, 2011, is presented below. The problems have been loosely categorized for convenience, although many of the problems could be listed under several different headings.

1. Multiplier sequences and CZDS

Problem 3.1. *Let $\{c_k\}_{k=0}^{n} \subset \mathbb{R}$ be given. Does there exist a real rooted polynomial, $p(x)$, with zeros outside $[0, n]$ such that $c_k = p(k)$, $k = 0, 1, \ldots, n$?*

Problem 3.2. *Let $f(z)$ be a meromorphic function. When is the sequence $\{f(k)\}_{k=0}^{\infty}$ a multiplier sequence?*

2. Matrix Theory

Problem .1. *Study the zeros of Bessis-Moussa-Villani polynomials $t \to tr(A + tB)^m$.*

Problem .2. *Find a simple proof of BMV conjecture for 3×3 matrices (and an explicit formula for the BMV measure for 3×3 matrices; the existence of the measure is due to recent work of Stahl).*

Problem .3. *Given $p \in \mathbb{C}[x]$, $\deg(p) = n$, find an $n\times n$ Hermitian matrix whose inertia (n_+, n_-, n_0) counts the zeros of p in $H_+, H_-, H_0 = \mathbb{R}$.*

Problem .4. *Any real polynomial can be written as $\det(A - \lambda I)$, where A has the (tridiagonal) form*

$$\Big[cccccc + + - 0 + \; - 0 + \; \ddots \ddots \ddots \; - 0 + \; - -$$

.

Problem .5. *Investigate the signs of Hurwitz minors for Pólya frequency sequences.*

3. Log Concavity

Let $\mathcal{L}$-$\mathcal{P}$ denote the class of functions which are locally uniform limits of polynomials having only real zeros, and let the class of functions in $\mathcal{L}$-$\mathcal{P}$ which have only non-negative Maclaurin coefficients be denoted $\mathcal{L}$-$\mathcal{P}^+$.

Problem .1. *Let*

$$Q_n^{(\alpha,\beta)} := \sum_{k=0}^{n} \binom{n}{k} f_k f_{n-k}((x+\alpha)_k (x+\beta)_{n-k} - (x+\alpha+\beta)_k (x)_{n-k}),$$

where $\alpha, \beta > 0$, $f_k^2 \geq f_{k-1} f_{k+1}$, *and* $(x)_k = x(x+1)\cdots(x+k-1)$. *Then the Maclaurin coefficients of* $Q^{(\alpha,\beta)}$ *are non-negative.*

Remark. [D. Karp] I have proved the case $\alpha = \beta = 1$ and formulated a number of generalizations of this conjecture in http://arxiv.org/abs/1203.1482

The following conjecture of (McNamara et al.) was proved by P. Brändén. Let $p(x) := \sum_{k=0}^{n} a_k x^k$, with $a_{-1} := a_{n+1} := 0$. If $p \in \mathcal{L}\text{-}\mathcal{P}^+$, then

$$\sum_{k=0}^{n} \begin{vmatrix} a_k & a_{k+1} \\ a_{k-1} & a_k \end{vmatrix} x^k \in \mathcal{L}\text{-}\mathcal{P}.$$

S. Fisk generalized this statement to polynomials with coefficients formed from the determinants of 3×3 matrices in the conjecture below.

Conjecture .2. *If* $p \in \mathcal{L}\text{-}\mathcal{P}^+$, *then*

$$\sum_{k=0}^{n} |ccc a_k a_{k+1} a_{k+2} a_{k-1} a_k a_{k+1} a_{k-2} a_{k-1} a_k$$

$x^k \in \mathcal{L}\text{-}\mathcal{P}^+$.

Problem .3. *Let* $\{\phi_k(x)\}_{k=0}^{\infty}$ *and* $\{f_k\}_{k=0}^{\infty}$ *be log-concave. When is* $\sum f_k \phi_k(x)$ *log-concave ?*

4. Combinatorics and Graph Theory

Let S_{2n} be the convex hull of all adjacency matrices of perfect matchings. Let

$$\mu_{2n} := \min\{\text{haf}(B) : B \in S_{2n}\},$$

where

$$\text{haf}(B) := \frac{1}{n!2^n} \frac{\partial}{\partial x_1 \cdots \partial x_{2n}} (x^T B x)^n$$

is the *hafnian* of B.

Problem .1.

(1) *Is it true that*

$$\alpha := \liminf_n \frac{\log \mu_{2n}}{2n} > -\infty \text{ ?}$$

(2) *Estimate* μ_{2n}.

The type-D_n Eulerian polynomials are given by

$$E(x) := \sum_{w} x^{\#\{s:\text{length}(ws)<\text{length}(w)\}}.$$

Problem .2. *Prove that $E(x)$ has only real zeros.*

Problem .3. *Is there a decent multivariate extension of the Chudnovsky-Seymour Theorem that the vertex independent set polynomial of a claw-free graph has only real roots?*

Problem .4. *Generalize the Chudnovsky-Seymour (Heilmann-Lieb) Theorem. Extend the common interlacing property to several variables.*

5. Zeros of derivatives

Problem .1. *Let $p \in \mathbb{R}[x]$, $\deg(p) = n > 2$. There are at least $n-1$ points of extreme curvature $\kappa(p)$ (more generally where $\kappa'(p) = 0$) [**?**].*

Problem .2. *What are the zero spacing effects of higher order differential operators on functions which belong to $\mathcal{L}$-$\mathcal{P}$?*

6. Orthogonal polynomials

Let (L_n^α) denote the generalized Laguerre polynomials. It is known that L_m^α and L_n^α can have common zeros if $|m-n| > 1$. Is the number of common zeros linked to arithmetic properties of α?

Conjecture .1. *There are no common zeros between L_m^α and L_n^α, $|m-n| > 1$, for α rational.*

Problem .2. *Let $P_n^{\alpha,\beta}$ denote the Jacobi polynomial of $\deg n$ with parameters $\alpha, \beta > -1$. Then $P_n^{\alpha',\beta'}$ and $P_n^{\alpha,\beta}$ interlace for $\alpha \approx \alpha'$, and $\beta \approx \beta'$. Does the failure of interlacing correspond to some physical law being "violated"?*

Problem .3. *Consider a subspace S of $\mathbb{C}[x]$ of codimension k (defined by k linear constraints, explicitly given). Given a scalar product on $\mathbb{C}[x]$, find an orthogonal basis of S (ordered by degree); some degrees will be missing.*

(1) *Which degrees get skipped?*
(2) *Characterize these orthogonal polynomials.*
(3) *Does this have anything to do with lacunary Padé approximants?*

7. Riemann Ξ-function

Let

$$\Xi(z) := \int_{-\infty}^{\infty} \Phi(t)\cos(zt)dt,$$

where

$$\Phi(t) := \sum_{n=1}^{\infty} (2n^4\pi^2 e^{9t} - 3n^2\pi e^{5t}) \exp(-n^2\pi e^{4t}).$$

Let $p_k(z)$ be the polynomials orthogonal with respect to Φ on $(-\infty, \infty)$. Define

$$P_n(z) := \frac{\det\left[p_i(z_j)\right]_{i,j=1}^n}{\det\left[z_i^{j-1}\right]_{i=1,j=1}^n}, \qquad \text{and} \qquad Q_n(z) := \frac{\det\left[p_i(z_j)\right]_{i,j=2}^n}{\det\left[z_i^{j-2}\right]_{i,j=2}^n}.$$

Problem .1. *Investigate the zeros of P_n and Q_n.*

Remark. [D. Dimitrov] The following generalization of the result mentioned above is true.

Let μ be an even, positive measure on $\mathbb{R}$, and $F(z) := \int_{-\infty}^{\infty} \cos(zt) d\mu(t)$. Let $\{p_k(z)\}_{k=0}^{\infty}$ be the orthogonal polynomials with respect to μ on $(-\infty, \infty)$. Then $F \in \mathcal{L}\text{-}\mathcal{P}$ if and only if the associated polynomial $P_n[p_1, \ldots, p_n]$ has purely imaginary zeros, if and only if $Q_n[p_1, \ldots, p_n]$ has no purely imaginary zeros. (for all n)

Problem 8.2. *Are there any non-real zeros of*

$$\int_0^{\infty} \Phi^{\alpha}(t) \cos(zt) dt \qquad \textit{for} \qquad \alpha > 0?$$

8. Zeros of specific functions

Conjecture .1. *Let*

$$p_n(z) = \sum_{k=0}^{n} (-1)^k (2k+1) z^{k(k+1)}$$

where, $n = 2, 3, 4, \ldots$. Each p_n has no real zeros for even n, and exactly two real zeros for odd n.

Problem .2. *Consider the function*

$$f(z) := \sum_k \frac{z^k}{(a^k+1)(a^{k-1}+1)\cdots(a+1)}, \qquad a > 1.$$

For which values of a is f in $\mathcal{L}\text{-}\mathcal{P}$?

Let

$$P_n(z, w) = \sum_{k=0}^{n} \binom{n}{k} z^k w^{k(n-k)}$$

Conjecture .3. *For all $|w| > 1$, all the zeros of $P_n(\cdot, w)$ are simple and separated in modulus (by a factor of at least $|w|^2$).*

Problem .4. *Generalize conjecture .3 to $Q_n(z, w) = \sum_{k=0}^{n} a_k z^k w^{k(n-k)}$. For what class of $\{a_n\}$ does the conjecture still hold?*

9. Miscellaneous Problems

Problem .1. *Let be $K \subset \mathbb{C}$ be compact, let $\mathbb{C} \setminus K$ be connected, and let f be a continuous function on K which is holomorphic and non-zero on the interior of K. Is there a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ with no zeros in K such that $p_n \to f$?*

Conjecture .2. *Let be $K \subset \mathbb{C}$ be compact, $\mathbb{C} \setminus K$ connected, and f be a continuous function on K which is holomorphic and non-zero on the interior of K. Suppose $K \subseteq \{z : \frac{1}{2} < \mathrm{Re} z < 1\}$, then*

$$\overline{d}\left(\{t \in \mathbb{R} : \|\zeta(\cdot + it) - f(\cdot)\|_{\infty,K} < \epsilon\}\right) > 0,$$

where

$$\overline{d}(E) := \limsup_{T\to\infty} \frac{m(E \cap [0, T])}{T}.$$

Remark. [P. Gauthier] Andersson has shown that a positive answer to Problem .1 and a confirmation of Conjecture .2 are equivalent.

Problem .3. *Let μ be a probability measure on $\{0, 1\}^n$. Consider*

$$f(\vec{z}) = \int \vec{z}^{\vec{\alpha}} d\mu(\vec{\alpha})$$

and

$$g(\vec{\lambda}) = \int e^{\vec{\lambda}\cdot\vec{\alpha}} d\mu(\vec{\alpha}).$$

What property of g is equivalent to the stability of f?

Problem .4. *Let $p(z) \in \mathbb{C}[z]$, $\deg(p) = n$. Suppose $p(z) \neq 0$ for all $|z| \leq 1$. Define $p^*(z) := z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$. Then it is known that*

$$p(A)p(A)^{\dagger} \geq p^*(A)(p^*(A))^{\dagger} \text{ for any contractive matrix } A.$$

(A is contractive *means $\sup_x \frac{\|Ax\|_2}{\|x\|_2} < 1$.) In 2D, we replace A with a pair of commuting contractions and p with a bivariate polynomial with no zeros in $\{z : |z| \leq 1\} \times \{w : |w| \leq 1\}$. The 3D generalization fails – when does it hold?*

Problem .5. *Find more examples and attempt to characterize Markov processes preserving stability. Explicitly, let $\psi : \mathbb{R}_{MA}[x, y, z] \to \mathbb{R}_{MA}[x, y, z]$ be a linear operator. When is $e^{t\psi} : \mathbb{R}[x, y, z] \to \mathbb{R}[x, y, z]$ stability preserving for all $t \geq 0$?*

References