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GROMOV-WITTEN INVARIANTS AND NUMBER THEORY

EDITED BY

Since the early 90's Gromov-Witten theory on Calabi-Yau threefolds has grown into a subject with impact on many branches of mathematics and physics. Spurred by its relevance, the mathematical foundation on Gromov-Witten invariants has been fully developed; the genus zero Gromov-Witten invariants are largely understood in mathematics, both theoretically and computationally. The theory of Gromov-Witten invariants of toric varieties was fully understood. Nevertheless Gromov-Witten theory at arbitrary genus on Calabi-Yau threefolds remains a daunting challenge to mathematicians as well as physicists. Mirror symmetry suggest to view the generating series of Gromov-Witten invariants globally as sections over the moduli of families of Calabi-Yau manifolds. The rigidity of the existing structures implied by this suggestion have raised high hopes for a breakthrough in Gromov-Witten theory in the near future. The new structures are centered around a generalization of the theory of quasi-modular forms, a classic object in number theory.

A concurrent development has been happening in number theory, in particular on Shimura varieties. Shimura varieties (e.g. modular curves) have naturally defined cycles. The generating functions of these cycles provide a very interesting way to organize and study these cycles. It was first studied by Hirzebruch and Zagier in 70s in their seminal work on intersections on Hilbert modular surfaces, which was then grandly extended by Kudla and Millson in 80's to Shimura varieties of orthogonal and unitary type. They proved that the generating functions of the 'special cycles' are modular forms if you look at their cohomology. In 90's Kudla, inspired by his 'so-called' Kudla program, conjectured that the generating functions are actually modular forms in Chow groups. In divisor case, this was proved by Borcherds in 1999, and the general cases are 'almost true' by Wei Zhang, and later Xinyin Yuan, Shou-Wu Zhang, and Wei Zhang. Kudla went a lot further, and defined Green functions for the divisors (the case for general cycles are still open), and conjectured that the generating functions are 'holomorphic' part of 'quasi-modular forms' (in number theory, we called them non-holomorphic modular forms, Harmonic weak Maass forms, ... depending on the occasions).

One of the most noticeable recent developments in theory of modular forms was inspired by Ramanujan's last letter to Hardy about Mock theta functions. Zagier, Zwegers, and Ono initiated in the early 2000's a study of the class of Mock modular forms, explaining Ramanujan's observation and finding many new examples of Mock modular forms. They, along with Kathrin Bringmann, and others, developed the foundations of the theory. Roughly speaking the key question is, given a holomorphic generating function

$$g(\tau) = \sum_{n=0}^{n} a_n q^n, \quad q = e^{2\pi\tau} ,$$

where $\tau = u + iv \in \mathbb{C}$ with $v > 0$, when and how to extend it to a 'non-holomorphic' modular form

$$f(\tau) = g(\tau) + \sum_{n \leq 0} a_n(v) q^n.$$

Zagier and collaborators have studied examples in which Mock modular forms occur as generating functions for Bogomol'nyi-Prasad-Sommerfield ground states, which are in simple situations directly related to generating functions for Gromov-Witten invariants. The current interactions of Gromov-Witten invariants and number theory is still in its early stage. But it has already touched on many deep aspects of both fields such as Gromov-Witten classes and cycle valued modular/automorphic forms.

1. HOLOMORPHIC ANOMALY EQUATION

Brief introduction to the holomorphic anomaly equation taken from Albrecht Klemm's Slides on on Omega backgrounds and generalized holomorphic anomaly equation at STring-math in June 2011. http://www.math.upenn.edu/StringMath2011/notes/Klemm$_StringMath2011_talk.pdf, SeealsothepaperbyBer

The B-model definition of the $F^g(a) = F^{(0,g)}(a)$ is given by

$$F^g(a) = \int_{\bar{\mathcal{M}}_g} \langle \prod_{k=1}^{3g-3} \beta^k \bar{\beta}^k \rangle_g \cdot [dm \wedge d\bar{m}],$$

The contraction of the coordinates $m_k,\ \bar{m}_k$ with the genus $g$worldsheet correlator of

$$\beta^k = \int_{\Sigma_g} G^- \mu^k, \quad \bar{\beta}^k = \int_{\Sigma_g} G^- \bar{\mu}^k$$

gives a real $6g-6$ form on the compactified moduli space $\bar{\mathcal{M}}_g$ of the g Riemann surface Σ_g.

An infinitessimal anholomorphic perturbation

$$\S(t_i, \bar{t}_i) = S(t_i) + \bar{(t)}^i \int_{\Sigma_g} \bar{O_i^{(2)}},$$

with

$$\bar{O_i^{(2)}} = \{G_0^+, [\bar{G}_0^+, \bar{O_i^{(0)}}]\} dz d\bar{z}$$

corresponds to an insertion of exact forms. The deformation receives contributions from the boundaries. This leads to the Holomorphic Anomaly Equation (Bershadski, Cecotti, Ooguri, and Vafa 1993)

$$\bar{\partial}_i F^g = \frac{1}{2} \bar{C}_i^{jk} (D_j D_k F^{g-1} + \sum_{h=0}^{g-1} D_j F^h D_k F^{g-h}), \quad g > 1$$

Defining for $g \geq 1$

$$F^{(n,g)}(t) = \int_{\bar{\mathcal{M}}_g} \langle O^n \prod_{k=1}^{3g-3} \beta^k \bar{\beta}^k \rangle_g \cdot [dm \wedge d\bar{m}],$$

and for $g = 0$

$$F^{(n+1,0)} = \langle \phi^{(0)}(0) \phi^{(0)}(1) \phi^{(0)}(\infty) O^n \rangle_{g=0}$$

where the field operator O should come from integration a 2-form over the Riemann surface, i.e.

$$O = \int_{\Sigma_g} \phi^{(2)},$$

and $\phi^{(2)}$ emerges as usual from the descending equation from $\phi^{(0)}$

Problem .1. *Does the holomorphic anomaly equation "integrate" over elliptic fibrations?*

Holomorphic Anomaly Equation

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Eta-products and root systems and holomorphic anomaly equation

As it turns out mock modular forms (such as eta-products) do not satisfy the holomorphic anaomaly equation.

Problem .2. *Do they satisfy other differential equations?*

Problem .3. *Investigate the Holomorphic anomaly equation (HAE) in fibered Calabi-Yau 3-folds and BPS counting.*

2. Jacobi forms

Problem .1. *What is a quasi-Jacobi form?*
What are interesting examples?
Generalize this notion to (quasi) automorphic forms.
Generalize this notion to mock Siegel modular forms.

Problem .2. *Is there a mock version of skew holomorphic Jacobi forms?*

Problem .3. *Are there connections between low index Mock modular Jacobi forms and finite groups (and geometry)?*

3. Calabi-Yau manifolds

Problem .1. *When can the elliptic genus be defined for noncompact Calabi-Yau manifolds?*

Problem .2. *Is there an index theoretic interpretation of elliptic genus? When it can, how do we compute it and what is the geometric interpretation of the coefficients? Check that this matches CFT calculations.*

Problem .3. *When can modular forms on Calabi Yau 3-folds moduli spaces be reduced to lower dimensional forms?*

Problem .4. *Investigate the role of paramodular groups in counting problems in Calabi Yau 3-folds*

4. Specific functions

Problem .1. *Can we calculate the leading terms of $F_g^B(q)$ at the orbifold point $z = 0$ of the quintic*

$$(\sum x_i^5 + z \prod x_i = 0)/\mathbb{Z}_5^3?$$

Here h_m is the m-th Hurwitz class number, i.e. the number of equivalence classes of positive definite binary quadratic forms of discriminant $-m$ with the class containing $x^2 + y^2$ weighted by 1/2 and the class containing $x^2 + xy + y^2$ weighted by 1/3. Moreover by convention $h_0 = -1/12$.

For more information in regards to the connection between physics and number theory, see the following work of

Kathrin Bringmann and Ben Kane http://arxiv.org/pdf/1305.0112v1.pdf
Katrin Bringmann and Sameer Murthy http://arxiv.org/pdf/1208.3476v2.pdf
Katrin Bringmann and Jan Manschot http://arxiv.org/pdf/1304.7208v1.pdf

Problem .2. *Do the coefficients of h_m have enumerative significance? Can we find h_m for $m \geq 9$?*

Problem .3. *Does $F = \sum f_g(\tau)\lambda^{2g-2}$ have transformation properties with respect to λ?*

Problem .4. *Is $\sum \frac{(5n)!}{(n!)^5} z^n$ related to (non-holomorphic) forms? Is it related to automorphic objects? Do other solutions to the Picard-Fuchs equation have modular properties?*

5. Other problems

Problem .1. *Do the L-series of mixed mock modular forms have interesting properties?*

Problem .2. *Do mixed mock modular forms satisfy differential equations with respect to a modular function?*

Problem .3. *Determine the connection between Mock modular forms and geometric invariants like Gromov-Witten, Donaldson, and OSV conjecture.*

References