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CLASSIFYING FUSION CATEGORIES

EDITED BY

1. General fusion category questions

Problem .1. *Is any integral fusion category unitarizable?*

Remark. [Cesar Galindo, Eric Rowell] By a recent paper of Galindo, Hong and Rowell, a slightly stronger statement can be proved for weakly group-theoretical categories: they are "completely unitary." Since weakly g.-t. categories conjecturally contain all weakly integral fusion categories, the answer is probably "yes."

Problem .2. *Is every integral fusion category weakly group theoretical?*

Problem .3. *Does pseudo-unitary imply unitarizable?*

Remark. [Wang] Physicists are really interested in this question. Given a conformal field theory which is not unitary, there is a negative dimension.

Problem .4. *Are all fusion categories pivotal? Are all fusion categories spherical? Does it depend on the ground field (k v.s. $\mathbb{C}$)?*

"Kaplanskys sixth conjecture" for fusion categories

Problem .5. *Is* $\frac{FPdim(C)}{FPdim(X)}$ *an algebraic integer for every* $X \in Irr(C)$*?*

2. How many fusion categories...

Problem .1. *Can we find all fusion categories with a given smallest simple object (which is not invertible)?*

Remark. [Scott Morrison] For example, we know that the smallest possible fusion dimension $1/2(\sqrt{3} + \sqrt{7})$ is realized by the Izumi-Xu-Ostrik fusion category from [**?**]. What other fusion categories contain an object with this dimension?

Problem .2. *Is there an effective version of Ocneanu rigidity? Is there a sub-exponential bound on the number of unitary fusion categories with respect to N, the sum of all the fusion multiplicities* $N_{i,j}^k$*?*

Problem .3. *How many fusion categories have the same given fusion rules?*

Problem .4. *What can you say about all fusion categories C for which* $\#\{\dim(X)|X \in C\} = 2$?

3. BRAIDINGS

Property F conjecture

If a fusion category has property F, then can't use braiding alone for a universal quantum computer.

Problem .1. *Given a braided, weakly integral fusion category, is the image of the braid group finite?*

Is braided and weakly integral fusion equivalent to finite image of the braid group?

Problem .2. *Is there a physical model which gives infinite image for the braid group?*

Problem .3. *For unitary theories, we can choose a gauge so the F matrices formed by the* $6j$ *symbols are unitary, and the braiding matrices are diagonal with respect to a certain basis. Can this happen for some non-unitary fusion category?*

Is the unitarity of the F matrices equivalent to unitarity?

4. NUMBER THEORETIC QUESTIONS

Problem .1. *Is there a sense in which a randomly chosen fusion graph doesn't have cylotomic dimensions?*

Problem .2. *Look at all spoke graphs with* $N > 0$ *arms. Are there finitely many N-tuples* $(\ell_1, \ldots, \ell_N)$ *such that the spoke graph with N arms of lengths* $\ell_1, \ldots, \ell_N$ *has cyclotomic norm squared?*

Problem . *Is there a positive real number which is a cyclotomic integer and is largest amongst its Galois conjugates, but which is not realized as the dimension of an object in a fusion category?*

Remark. [Scott Morrison] The first five such numbers above 2, given in [**?**], are all known to be realized.

5. OBJECTS IN FUSION CATEGORIES

Problem .1. *What values can* $\dim(X)$ *take in* $[2, 3]$ *for* $X \in C$*, a braided fusion category?*

Remark. These two dimensions come from Cyclotomic integers, fusion categories, and subfactors Frank Calegari, Scott Morrison and Noah Snyder, Communications in Mathematical Physics Volume 303, Issue 3 (2011), pp. 845-896 [**?**].

Problem .2. *What are all the* C *generated by* X *with* $FPdim(X) \leq 2$*, and* X *not self-dual? Are they group theoretical if* $\dim(X) = 2$*?*

A fusion category version of supertransitivity

Problem .3. *In a fusion category, is there an upper bound on the* N *such that* $X^{\otimes N}$ *is a simple object (where* $\dim(X) > 1$*)?*

Problem .4. *There are accumulation points from below for* $FPdim(X)$ *for an object in a fusion category or* $[M : N]$ *for finite depth subfactors. Are there any accumulation points from above?*

Remark.[Scott Morrison] Are there non-integer accumulation points?

6. Frobenius-Schur indicators

Problem .1. *Is there a way to find the Frobenius-Schur exponent of* C *without computing* $Z(C)$*?*

7. Examples

Problem .1. *Classify module categories and Brauer-Picard groups for known examples.*

8. Subfactors

The supertransitivity with respect to an object X with $\dim(X) > 2$ is the largest N such that $\mathrm{Hom}(1, X^{\otimes n})$ is Temperley-Lieb.

Supertransitivity is the analog of transitivity of group actions.

Problem .1. *Is there an upper bound on the supertransitivity of a subfactor planar algebra?*

Remark.[Scott Morrison] Currently the extended Haagerup subfactor holds the record, with $n = 7$. The Asaeda-Haagerup subfactor has $n = 5$, and otherwise all known examples have $n \leq 4$.

Problem .2. *Find a non-number theoretic argument to rule out the rest of the Haagerup family vine.*

For example, is there a diagram that evaluates in two different ways?

Problem .3. *Is there a polymer theory of principal graphs? What graphs can appear as subgraphs of principal graphs?*

9. Extensions

Problem .1. *Is there an extension theory for fusion categories extended by fusion rings? (For example, near group categories)*

10. Tensor functors

Problem .1. *Describe functors between group-theoretical categories.*
What is known for Verlinde categories? (quantum groups at roots of unity)

11. Fusion categories from subfactors

Problem .1. *Can you classify all algebras in fusion categories with small Frobenius-Perron dimension (e.g., less than* $3 + \sqrt{3}$*)?*

Problem .2. *Compute the center of the even half of the Asaeda-Haagerup and extended Haagerup subfactors.*

Problem .2. *Is there a conceptual construction of the even half of 4442?*

12. Finite tensor categories

Problem .1. *Suppose* C *is a finite tensor category over* $\mathbb{C}$ *with prime Frobenius-Perron dimension. Is* C *fusion?*
(Hence, it would be of the form Vect$(\mathbb{Z}/p, \omega)$*. This would be an extension of a result in Hopf algebras.)*

Problem .2. *How much from modular representations of finite groups can be carried to finite tensor categories?*

References